

Beltrami–Bernoulli Equilibria in Plasmas with Degenerate Electrons

V. I. Berezhiani*

*Andronikashvili Institute of Physics, TSU, Tbilisi 0177, Georgia and
School of Physics, Free University of Tbilisi, Georgia*

N. L. Shatashvili†

*Andronikashvili Institute of Physics, TSU, Tbilisi 0177, Georgia and
Department of Physics, Faculty of Exact and Natural Sciences,
Ivane Javakhishvili Tbilisi State University (TSU), Tbilisi 0179, Georgia*

S. M. Mahajan‡

Institute for Fusion Studies, The University of Texas at Austin, Austin, Tx 78712

A new class of Double Beltrami–Bernoulli equilibria, sustained by electron degeneracy pressure, are investigated. It is shown that due to electron degeneracy, a nontrivial Beltrami–Bernoulli equilibrium state is possible even for a zero temperature plasma. These states are, conceptually, studied to show the existence of new energy transformation pathways converting, for instance, the degeneracy energy into fluid kinetic energy. Such states may be of relevance to compact astrophysical objects like white dwarfs, neutron stars etc.

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I. INTRODUCTION

Constrained minimization of fluid energy with appropriate helicity invariants has provided a variety of extremely interesting equilibrium configurations that have been exploited and found useful for understanding laboratory as well as astrophysical plasma systems (see e.g. [1–6] and references therein). Two particularly simple manifestations of this genre of equilibria (called Beltrami states) are: 1) The single Beltrami state, $\nabla \times \mathbf{B} = \alpha \mathbf{B}$, discussed by Woltejr and Taylor [2, 3] in the context of force free single fluid magnetohydrodynamics (MHD), and 2) a more general Double Beltrami State accessible to Hall MHD – a two-fluid system of ions and inertialess electrons [7]; the latter has been investigated, in depth, by Mahajan and co-workers [8–14]. The Beltrami condition implies an alignment of the fluid vorticity and its velocity, and the characteristic number of a state is determined by the number of independent single Beltrami systems needed to construct it.

The Beltrami conditions must be buttressed by an appropriate Bernoulli constraint to fully describe an equilibrium state; it is, then, more descriptive to call them Beltrami–Bernoulli (BB) states.

Although the BB class of equilibria have been studied for both relativistic and non-relativistic plasmas, most investigations are limited to what may be termed “dilute” or non-degenerate plasmas so that the constituent particles are assumed to obey the classical Maxwell-Boltzman statistics. It is natural to enquire how such states would

change/transform if the plasmas were highly dense and degenerate (the mean inter-particle distance is smaller than the de Broglie thermal wavelength) so that their energy distribution was dictated by Fermi-Dirac statistics. Notice that at very high densities, the particle Fermi Energy can become relativistic and the degeneracy pressure may dominate the thermal pressure.

Such highly dense/degenerate plasmas are found in several astrophysical and cosmological environments as well as in the laboratories devoted to inertial confinement and high energy density physics; in the latter intense lasers are employed to create such extreme conditions [15–18]. Compact astrophysical objects like white and brown dwarfs, neutron stars, magnetars with believed characteristic electron number densities $\sim 10^{26} - 10^{32} \text{ cm}^{-3}$, formed under extreme conditions, are the natural habitats for dense/degenerate matter [19–25].

In this paper, we develop the simplest model in which the effect of quantum degeneracy on the nature of the BB class of equilibrium states can be illustrated. Emphasizing the quantum degeneracy effects, the aim of this paper is complementary to that of [26, 27] in which the BB like states of a neutral fluid are investigated when another quantum phenomenon – the spin vorticity – plays a fundamental role. We will choose a model hypothetical system (later we would show its relevance to specific aspects of a white dwarf (WD)) of a two-species neutral plasma with non-degenerate non relativistic ions, and degenerate relativistic electrons embedded in a magnetic field. To make the model conform closely to the standard Double BB system, it will be further assumed that, despite the relativistic mass increase, the electron fluid vorticity is negligible compared to the electron cyclotron frequency (such a situation may pertain, for example, in the pre-WD state of star evolution, and in the dynamics of the

*Electronic address: vazhab@yahoo.com

†Electronic address: shatash@ictp.it

‡Electronic address: mahajan@mail.utexas.edu

WD atmosphere). The study of the degenerate electron inertia effects on the Beltrami States in dense neutral plasmas will constitute the scope of a future publication.

II. MODEL

For an ideal isotropic degenerate Fermi gas of electrons at temperature T_e , the relevant thermodynamic quantities – the pressure \mathcal{P}_e and the proper internal energy density \mathcal{E}_e (the corresponding enthalpy, $w_e = \mathcal{E}_e + \mathcal{P}_e$) per unit volume – can be calculated to be [28, 29]

$$\mathcal{P}_e = \frac{m_e^4 c^5}{3\pi^2 \hbar^3} f(P_F), \quad (1)$$

$$\mathcal{E}_e = \frac{m_e^4 c^5}{3\pi^2 \hbar^3} \left[P_F^3 \left((1 + P_F^2)^{1/2} - 1 \right) - f(P_F) \right], \quad (2)$$

where

$$8f(P_F) = 3 \sinh^{-1} P_F + P_F (1 + P_F^2)^{1/2} (2P_F^2 - 3) \quad (3)$$

and $P_F = p_F/m_e c$ is the normalized Fermi momentum of electrons; the Fermi energy may be expressed in terms of P_F as $\epsilon_F = m_e c^2 \left[(1 + P_F^2)^{1/2} - 1 \right]$. It is useful to note that p_F is related to the rest-frame electron density n_e via $p_F = m_e c (n_e/n_c)^{1/3}$, where $n_c = 5.9 \times 10^{29} \text{ cm}^{-3}$ is the critical number-density at which the Fermi momentum equals $m_e c$ [30], and defines the onset of the relativistic regime. The electron plasma is treated as the completely degenerate gas – the thermal energy of electrons is much lower than their Fermi energy ($n_e T_e / \mathcal{P}_e \ll 1$). The distribution function of electrons remains locally Juttner-Fermian which for zero temperature case leads to the just density dependent thermodynamical quantities $\mathcal{E}_e(n_e)$, $\mathcal{P}_e(n_e)$ and $w_e(n_e)$. All these quantities implicitly depend on space-time coordinates via $n_e = N_e / \gamma_e$, where N_e is the density in laboratory frame of the electron-fluid; $\gamma_e = (1 - V_e^2/c^2)^{-1/2}$ is the Lorentz factor. The electron plasma dynamics is isentropic and, consequently, obeys the thermodynamical relation $d(w_e/n_e) = (d\mathcal{P}_e)/n_e$. Applying this relation, and after straightforward algebra (see e.g. [31–33]), the equation of motion for degenerate electron fluid reduces to:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\sqrt{1 + P_F^2} \mathbf{p}_e \right) + m_e c^2 \nabla \left(\sqrt{1 + P_F^2} \gamma_e \right) = \\ = -e\mathbf{E} - \frac{e}{c} \mathbf{V}_e \times \mathbf{B} + \frac{e}{c} \mathbf{V}_e \times \nabla \times \left(\sqrt{1 + P_F^2} \mathbf{p}_e \right) \end{aligned} \quad (4)$$

with $\mathbf{p}_e = \gamma_e m_e \mathbf{V}_e$ being electron hydrodynamical momentum and under our assumption of negligible electron fluid vorticity the last term can be negligible. For the

non-degenerate ion fluid we have the equation of motion written as (m_i is a proton mass):

$$\begin{aligned} m_i \left[\frac{\partial \mathbf{V}_i}{\partial t} + (\mathbf{V}_i \cdot \nabla) \mathbf{V}_i \right] = -\frac{1}{N_i} \nabla p_i + \\ + e\mathbf{E} + \frac{e}{c} \mathbf{V}_i \times \mathbf{B}. \end{aligned} \quad (5)$$

Since this short paper is devoted to bringing out the simplest effects of electron degeneracy on BB states (that may be very useful in understanding some aspects of the appropriate astrophysical objects and their evolution), we will borrow verbatim most of the results for the electron and ion dynamics [7, 9, 10]. For non relativistic ions, and inertialess electrons, there are two independent Beltrami conditions (aligning the electron and ion generalized vorticities along their respective velocities):

$$\mathbf{b} = a N \left[\mathbf{V} - \frac{1}{N} \nabla \times \mathbf{b} \right], \quad (6)$$

$$\mathbf{b} + \nabla \times \mathbf{V} = d N \mathbf{V}, \quad (7)$$

where $\mathbf{b} = e\mathbf{B}/m_i c$ and it was assumed, that electron and proton densities are nearly equal - $N_e \simeq N_i = N$; here a and d are dimensionless constants related to the two invariants: the magnetic helicity $h_1 = \int (\mathbf{A} \cdot \mathbf{b}) d^3x$ and the generalized helicity $h_2 = \int (\mathbf{A} + \mathbf{V}) \cdot (\mathbf{b} + \nabla \times \mathbf{V}) d^3x$ of the system; here \mathbf{A} is the dimensionless vector potential.

Notice that, following the conventional treatments, we have written our equations in terms of normalized one fluid variables: the fluid velocity \mathbf{V} and the current $\mathbf{J} = \nabla \times \mathbf{b}$ (via Ampere's law) in terms of which, the electron and the ion speeds are given by $\mathbf{V}_e = \mathbf{V} - (1/N) \nabla \times \mathbf{b}$, and $\mathbf{V}_i = \mathbf{V}$, respectively (the electrons are assumed to be inertia less). In this approximation of inertia less electrons, the electron vorticity is primarily magnetic (\mathbf{b}) while the ion vorticity has both kinematic and magnetic parts ($\mathbf{b} + \nabla \times \mathbf{V}$).

In the preceding equations, the density is normalized to N_0 (the corresponding rest-frame density is n_0); the magnetic field is normalized to some ambient measure B_0 ; all velocities are measured in terms of the corresponding Alfvén speed $V_A = B_0 / \sqrt{4\pi N_0 m_i}$; all lengths [times] are normalized to the skin depth λ_i [λ_i / V_A], where $\lambda_i = c / \omega_{pi} = c \sqrt{m_i / 4\pi N_0 e^2}$.

As mentioned in the introduction, the Beltrami conditions (6) and (7) must be supplemented by the Bernoulli constraint to define an equilibrium state (the stationary solution of the dynamical system). In the present context, the constraint reads as

$$\nabla \left(\beta_0 \ln N + \mu_0 \sqrt{1 + P_F^2} \gamma + \frac{V^2}{2} \right) = 0 \quad (8)$$

where β_0 is the ratio of the thermal pressure to the magnetic pressure, and $\mu_0 = m_e c^2 / m_i V_A^2$ and for the electron fluid Lorentz factor we put $\gamma_e \simeq \gamma(\mathbf{V})$. Stated equivalently, Bernoulli condition (8) is an expression of the balance of all the remaining potential forces when Beltrami conditions (6), (7) are imposed on the two-fluid equilibrium equations.

Since $P_F = p_F / m_e c = (N N_0 / n_e \gamma)^{1/3}$ $[= (N n_0 / n_e)^{1/3}]$ is a function of the density N , the system of equations (6)-(7)-(8) forms a fully specified equilibrium – a complete system to determine \mathbf{N} , \mathbf{V} , and \mathbf{b} . Notice that the equilibrium continuity equation $[\nabla \cdot (N\mathbf{V}) = 0]$ and the divergence free condition for magnetic field $[\nabla \cdot \mathbf{b} = 0]$ are automatically satisfied. The simplest double BB equilibrium configuration in plasmas with degenerate electrons has following noteworthy features:

1) The Beltrami conditions reflect the simple physics: (i) the inertia-less (despite the relativistic increase in mass) degenerate electrons follow the field lines, (ii) while the ions, due to their finite inertia, follow the magnetic field modified by the fluid vorticity. The combined field $\mathbf{b} + \nabla \times \mathbf{V}$, an expression of magneto-fluid unification, may be seen either as an effective magnetic field or an effective vorticity.

2) The Beltrami conditions (6) and (7) are not directly affected by the degeneracy effects in the current approximation neglecting the electron inertia. In fact, these are precisely the two conditions that define the Hall MHD states. In the highest density regimes, however, the Fermi momentum (and hence the Lorentz factor $\gamma(\mathbf{V})$) may be so large that the effective electron inertia will have to be included in (6), the electron Beltrami condition.

3) In this minimal model, the electron degeneracy manifests, explicitly, only through the Bernoulli condition (8). The degeneracy induced term, proportional to μ_0 would go to unity (whose gradient is zero), and would disappear in the absence of the degeneracy pressure. For significant P_F , on the other hand, the degeneracy pressure can be far bigger than the thermal pressure (measured by β_0). In fact, the degenerate electron gas, can sustain a qualitatively new state: a nontrivial Double Beltrami–Bernoulli equilibrium at zero temperature. In the classical zero-beta plasmas, only the relatively trivial, single Beltrami states are accessible [34].

4) It is trivial to eliminate \mathbf{b} in Eqs. (6) and (7) to obtain

$$\frac{1}{N} \nabla \times \nabla \times \mathbf{V} + \nabla \times \left[\left(\frac{1}{aN} - d \right) N \mathbf{V} \right] + \left(1 - \frac{d}{a} \right) \mathbf{V} = 0, \quad (9)$$

which, coupled with (8), provides us with a closed system of four equations in four variables (N, \mathbf{V}) . Once this is solved with appropriate boundary conditions, one can invoke (7) to calculate \mathbf{b} . The reader can find the solution for the similar mathematical problem relevant to the non-degenerate case (in the context of solar atmosphere) in [9].

5) The Bernoulli condition (8) introduces a brand new player in the equilibrium balance; the spatial variation in the electron degeneracy energy (proportional to μ_0) could increase or decrease the plasma β_0 or the fluid kinetic energy (measured by V^2) in the corresponding region. Thus Fermi energy could be converted to kinetic energy; it could also forge a re-adjustment of the kinetic energy from a high-density/low-velocity plasma to a low-density/high-velocity plasma. Similar energy transformations, mediated through classical gravity, were discussed in Mahajan *et al* (2002; 2006).

The extensions as well as a detailed analysis of (8-9) are under investigation. For instance, when electron fluid degeneracy is very high and one can not neglect inertia effects in their generalized vorticity, the order of BB states is likely to rise; such higher order states (like the triple BB state when electron inertia is retained) have been studied for specific cases [35–38]. Another natural extension for the current formalism (super-relativistic electrons) will be the introduction of Gravity, which, in principle, could balance the highly degenerate electron fluid pressure. Gravity (Newtonian) effects in the BB system have been investigated in the solar physics context (e.g Mahajan *et.al* (2002), Mahajan *et. al.* (2005,2006); for disk-jet structure formation – [39, 40]).

Since our aim in this paper is merely demonstrating the possibility of Beltrami–Bernoulli equilibria sustained by electron degeneracy pressure, we will not work out the detailed solutions of Eqs. (8-9). Because this system, in its non-degenerate form, has been highly studied ([7, 9, 10, 14, 41] and references therein), we can safely draw interesting inferences about:

1) Some distinguishing features of the expected "degeneracy-modified" solutions and even the significance and possible applications of somewhat straightforward extensions of these solutions (keeping electron inertia and adding gravity, for example). Several of these general features have already discussed.

2) possible physical systems where such equilibrium solutions may find relevance.

A possible application of the "degenerate" BB states may be found in stellar physics. Here is a short summary of the relevant phenomenology:

It is well-known that when a star collapses, and cools down, the density of lighter elements increases affecting the total pressure/enthalpy of unit fluid element – first order departure from the classical e-i plasma; beyond the hot, pre-white dwarf stage, photon cooling dominates and gravitational contraction is dramatically reduced as the

interior equation of state hardens into that of a strongly degenerate electron gas. The degenerate electrons provide the dominant pressure, while the contribution of thermal motion of ions into the pressure is negligible (see the review [42] and references therein).

Recent studies show that a significant fraction of White Dwarfs are found to be magnetic with typical fields strengths below 1KG. Massive and cool White Dwarfs, interestingly, are found with much higher fields detected (see [43] and references therein). It is argued that the origin of magnetic fields in WD stars may be linked to possible field-generating merger events preceding the birth of the white dwarf. On the other hand, Wegg & Phinney (2012) concluded that the kinematics of massive WDs are consistent with the majority being formed from single star evolution. In [43] it was shown that WD stars with such surface temperatures that convection zones develop, seems to show stronger magnetic fields than hotter stars; the mean mass of magnetic stars seems to be on average larger than the mean mass of non-magnetic WD stars. Recent investigations ([45] and references therein) have uncovered several cool, magnetic, polluted hydrogen atmosphere (DAs) white dwarfs. It was found that the incidence of magnetism in old, polluted white dwarfs (DAZ) significantly exceeds what is found in the general white dwarf population suggesting a hypothetical link between a crowded planetary system and magnetic field generation. Polluted white dwarfs provide an opportunity to investigate the ultimate fate of planetary systems and, hence, it is of crucial importance to study the origin and evolution of surface magnetic fields of such DAZ-es.

Let us now explore, through a simple example, if degenerate BB states could shed some light on the physics of WDs. Considering High magnetic field white dwarfs, we assume: the degenerate electrons densities $\sim (10^{25} - 10^{29}) \text{ cm}^{-3}$; magnetic fields $\sim (10^5 - 10^9) \text{ G}$, and temperatures $\sim (40000 - 6000) \text{ K}$. For these parameters, the Alfvén speed $V_A \sim (10^4 - 10^6) \text{ cm/s}$, yielding $\beta_0 \sim (10^6 - 10^0)$ and $\mu_0 \sim (10^{10} - 10^6) \gg 1$. The ion skin-depth $\lambda_i \sim (10^{-5} - 10^{-7}) \text{ cm}$ turns out to be rather short.

For this class of systems, the second term (degeneracy pressure) in Eq.(8) is always much larger than the first term, the thermal pressure. Neglecting the first term, and remembering that for non relativistic flows (essential at ion speeds) $\gamma(\mathbf{V}) \sim 1$, Eq.(8) – Bernoulli Condition – with inclusion of classical (Newtonian) gravity (justified by observations for WDs) implies

$$\mu_0 \sqrt{1 + P_F^2} - \frac{R_A}{R} + \frac{V^2}{2} = \text{const} \quad (10)$$

where the *const* measures, in some sense, the main energy content of the fluid; the Beltrami conditions (6), (7) remain the same; R is a radial distance from the center of WD normalized to its radius $R_W [\sim (0.008 - 0.02) R_\odot]$ and $R_A = GM_W/R_W V_A^2$ (here G is the gravitational constant and M_W - WD mass). Since P_F is a function of

Fermi energy (and hence, of density), we assume that at some distance R_* (corresponding to density maximum), P_F reaches its maximum value P_{F*} . Taking the corresponding minimum velocity to be zero ($V_* \sim 0$), we find $\text{const} = \mu_0 \sqrt{1 + P_{F*}^2} - R_A/R_*$. The magnitude of the velocity is now determined to be

$$|\mathbf{V}| \sim \sqrt{2\mu_0} \kappa(P_F) \quad (11)$$

with

$$\kappa(P_F) = \left[\left(\sqrt{1 + P_{F*}^2} - \sqrt{1 + P_F^2} \right) - \frac{R_A}{\mu_0} \left(\frac{1}{R_*} - \frac{1}{R} \right) \right]^{1/2}. \quad (12)$$

Notice that the dimensionless coefficient $R_A/\mu_0 \ll 1$ measures the relative strength of gravity versus the degenerate pressure term. For WD-s with Mass $M_W \sim (0.8 - 0.25) M_\odot$ and radius $R_W \sim (0.013 - 0.02) R_\odot$, $R_A/\mu_0 \sim (0.2 - 0.04) \ll 1$; less massive the WD, the smaller is the coefficient.

In addition, the DB structure scales are small compared to R_W in outer layers of the WD (where our model applies). The gravity contribution to the flow velocity (at specific distance of outer layers of WD-s with $R \geq R_*$ and $(R - R_*)/R_* \ll 1$, $R_* \leq 1$), therefore, can be readily neglected. The gravity contribution, exactly like in the solar case [12, 13], determines the radial distance in WD's outer layer over which the "catastrophic" (fast) acceleration of flow may appear (due to the magneto-fluid coupling). In the regions where the flows are insignificant (at very short distances from the WD's surface) gravity controls the stratification but as we approach the flow "blow-up" distances (the flow becomes strong) the self-consistent magneto-Bernoulli processes take over and control the density (and hence the velocity) stratification.

Calculating the maximum flow velocity, occurring at $\kappa(P_F)$ maximum (density minimum), needs a detailed knowledge of the system. One does, however, notice that if $\sqrt{2\mu_0} \kappa(P_F) > 1$ the generated flow is locally super-Alfvénic in contradistinction to the non-degenerate, thermal pressure dominated plasma, when the maximal velocity due to the magneto-Bernoulli mechanism be locally sub-Alfvénic when local plasma beta < 1 as in the Solar Atmosphere. This simple example shows that the electron degeneracy effects can be both strong, and lead to interesting predictions like the anticorrelation between the density and flow speeds.

The richness introduced by electron-degeneracy to the the Beltrami-Bernoulli states could help us better understand compact astrophysical objects. When the star contracts, for example, its outer layers keep the multi-structure character although density in the structures becomes defined by electron degeneracy pressure. Then, important conclusion for future studies is that when studying the evolution of the atmospheres/outer layers of compact objects, flow effects can not be ignored. More specifically, the knowledge of the effects introduced by

flows (observed in stellar outer layers) acquired for classical plasmas can be used when investigating the dynamics of White Dwarfs and their evolution.

III. CONCLUSIONS

In conclusion, in the present paper we found the possibility of the existence of Double Beltrami relaxed states in plasmas with degenerate electrons (often met in astrophysical conditions). Since non degenerate double BB states guarantee scale separation phenomenon that, for appropriate conditions, provide energy transformation pathways for various astrophysical phenomena (eruptions, fast/transient outflow and jet formation, magnetic field generation, structure formation, heating and etc.), such pathways could be easily explored for the degener-

ate case with degeneracy pressure providing an additional energy source. Particularly interesting could be finding the fate of a Star, when contracting and cooling, and becoming a White Dwarfs since the latter is assumed to be a boundary condition for Stellar Evolution. Our future studies will be devoted to detailed investigation of present model to explain the existence of large-scale structures (like surface magnetic fields, flows and outflows, eruptions) in astrophysical objects with degenerate plasmas as well as to explore the evolution of multi-structure stellar outer layers when contracting, cooling.

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